Application of the Homotopy Analysis Method for Solving Equal-Width Wave and Modified Equal-Width Wave Equations

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Although the homotopy analysis method (HAM) is, by now, a well-known analytic method for handling functional equations, there is no general proof of its applicability to all kinds of equations. In this paper, by using this method to solve equal-width wave (EW) and modified equal-width wave (MEW) equations, we have made a new contribution to this field of research. Our goal is to emphasize on two points: one is the efficiency of HAM in handling these important family of equations and its superiority over other analytic methods like homotopy perturbation method (HPM), variational iteration method (VIM), and Adomian decomposition method (ADM). The other point is that although the considered two equations have different nonlinear terms, we have used the same auxiliary elements to solve them.

Key words: Homotopy Analysis Method; Equal-Width Wave Equation; Modified Equal-Width Wave Equation; Nonlinear Equations.
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1. Introduction

Nonlinear equations have been of considerable interest among scientists because they model many scientific problems arising in different fields. Although, numerical methods have largely been applied to solve these equations, some restrictions of numerical methods have motivated scientists to search for analytic solutions. So there are, also, analytic techniques for solving nonlinear problems, among the classic ones we can consider Lyapunov's artificial small parameter method, perturbation techniques, and the δ-expansion method. Recently the idea of giving analytic approximations to functional equations has largely developed and some new techniques have been proposed. Among the newly developed ones are the Adomian decomposition method (ADM), the homotopy analysis method (HAM), the variational iteration method (VIM), the homotopy perturbation method (HPM), the tanh method, the sine-cosine method, and the exp-function method.

Homotopy analysis method (HAM), first proposed by Liao [1, 2], is an elegant method which has proved its effectiveness and efficiency in solving many types of functional equations, see [1 – 13] and the references therein. In this method, Liao uses homotopy, a concept from topology, to continuously deform the nonlinear equation under study to a system of linear ones.

The reasons for the preference of HAM over other analytic techniques can be listed as follows:

(i) HAM properly overcomes restrictions of perturbation techniques because it doesn't need any small or large parameters to be contained in the equation.

(ii) Liao, in his book [2], proves that this method is a generalization of some previously used techniques such as δ-expansion method, artificial small parameter method, and ADM. Also, it is shown that HPM [14] is just a special case of HAM [10, 13, 15].

(iii) Unlike previous analytic techniques, HAM provides us with a convenient way to adjust and control the convergence region and rate of approximation series. This is done by using the so called $\hbar$-curves, see [2].

The present work concerns with the application of HAM in solving the equal-width wave (EW) equation and a modified form of this equation. We focus on two points: firstly, the applicability and efficiency of HAM in handling this family of wave-type equations and its superiority in comparison with other techniques.

Secondly, although the wave-type equations may have different nonlinear terms, when applying HAM we can treat them by using a general choice of auxiliary elements.

2. Basic Idea of HAM

For a good understanding of HAM the reader is referred to Liao's book [2], also a general introduction and recent developments could be found in [16]. In this section we briefly review the basic idea of HAM. Let us consider the following nonlinear equation in a general form:

$$N[u(r,t)] = 0,$$

where N is a nonlinear operator, $u(r,t)$ is an unknown function, and r and t denote spatial and temporal independent variables, respectively. At this stage we ignore initial or boundary conditions, because they are not used, directly, in construction of the so-called 'homotopy equation'. Actually we have to refer to these conditions for solving the resulting linear system, for more details see [2, 16, 17]. By means of generalizing the traditional homotopy method, Liao constructs the so-called 'zeroth-order deformation equation'

$$(1-q)L[\phi(r,t;q)-v_0(r,t)] = q\hbar H(r,t)N[\phi(r,t;q)], \quad (1)$$

where $q \in [0,1]$ is the embedding (or homotopy) parameter, $\hbar \neq 0$ is a non-zero auxiliary parameter – which recently renamed to 'convergence-control parameter', $H(r,t)$ is an auxiliary function, L is an auxiliary linear operator, $v_0(r,t)$ is an initial guess for $u(r,t)$, and $\phi(r,t;q)$ is an unknown function. It is important that we have great freedom to choose auxiliary elements in HAM [2]. Obviously, when $q = 0$ and $q = 1$ one has

$$\phi(r,t;0) = v_0(r,t), \quad \phi(r,t;1) = u(r,t).$$

Thus as q increases from 0 to 1, the solution $\phi(r,t;q)$ varies from the initial guess $v_0(r,t)$ to the solution $u(r,t)$. Expanding $\phi(r,t;q)$ in Taylor series with respect to q, we have

$$\phi(r,t;q) = u_0(r,t) + \sum_{m=1}^{\infty} u_m(r,t)q^m, \quad (2)$$

where

$$u_m(r,t) = \frac{1}{m!}\frac{\partial^m \phi(r,t;q)}{\partial q^m}|_{q=0}. \quad (3)$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter, and the auxiliary function are properly chosen, the series (2) converges at $q = 1$ and we have

$$u(r,t) = u_0(r,t) + \sum_{m=1}^{\infty} u_m(r,t), \quad (4)$$

which must be a solution to the original nonlinear equation, this has been proved by Liao [2]. If we set $\hbar = -1$ and $H(r,t) = 1$, equation (1) becomes

$$(1-q)L[\phi(r,t;q) - v_0(r,t)] + qN[\phi(r,t;q)] = 0,$$

which is a special case and is used in the homotopy perturbation method (HPM).

According to (3), the governing equations for $u_m(x,t)$ can be deduced from the zeroth-order deformation equation (1). Define the vector

$$\vec{u}_n = [u_0(r,t), u_1(r,t), \ldots, u_n(r,t)].$$

Differentiating equation (1) m times with respect to the embedding parameter q, then setting $q = 0$ and finally dividing by $m!$, we have the so-called 'mth-order deformation equation',

$$L[u_m(r,t) - \chi_m u_{m-1}(r,t)] = \hbar H(r,t)R_m(\vec{u}_{m-1}, r,t), \quad (5)$$

where

$$R_m(\vec{u}_{m-1}, r,t) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\phi(r,t;q)]}{\partial q^{m-1}}|_{q=0}, \quad (6)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1. \end{cases}$$

Directly substituting the series (2) into the zeroth-order deformation equation (1), equating coefficients of like-power of the embedding parameter q, one can get the same high-order deformation (5), as proved in [10, 15, 16]. Equation (5) is a linear one, so we have transformed a nonlinear equation to a set of linear ones, which can be easily solved using an iterative procedure. After solving (5), we can substitute $u_m(r,t)$ in (4) and obtain an approximation of arbitrary order. It worths noting that HAM, in many cases, leads to a series which gives the exact solution, see e. g. [17], and using the auxiliary parameter $\hbar$ one can adjust and control the convergence region and rate of approximation series. The discussion on how to use $\hbar$ is made in [2].

When we are aiming to get approximate solutions, we can use only the few first terms in (4), without having concern to get an exact solution series.

In HAM what we are mainly concerned about is how to choose the initial guess (v_0), the auxiliary linear operator L, and the auxiliary function $H(r,t)$. Liao, in his book [2], has proposed three rules for choosing these components, actually here we get advantage of all informations about the equation such as its physical background and boundary/initial conditions. In the cases, like our considered equations, it's common to choose the auxiliary function to be $H(r,t) = 1$, as used by many authors [7, 9, 17]. When we don't have enough information to decide on a set of base functions, this choice for H seems to be a good alternative.

It seems that when we don't emphasize on a set of base function, for representing the solution, there is no difference between HAM and HPM. This isn't true because the HAM still contains the convergence-control parameter $\hbar$ which provides us with a simple way to adjust and control the convergence region and rate of the approximation series. This advantage of HAM makes it superior in comparison with other analytic methods and perturbation techniques.

3. Equal-Width Wave (EW) Equation

The equal-width wave (EW) equation plays a major role in the study of nonlinear dispersive waves since it describes a broad class of physical phenomena such as shallow water waves and ion acoustic plasma waves. The EW equation, derived for long waves propagating in the positive x-direction has the form

$$u_t + uu_x - u_{xxt} = 0, \tag{7}$$

with the initial condition

$$u(x,0) = 3\ \mathrm{sech}^2(\frac{x-15}{2}). \tag{8}$$

In the fluid problem u is related to the vertical displacement of the water surface, while in the plasma application u is the negative of the electrostatic potential.

A numerical simulation and explicit solution of the EW equation were obtained by Raslan [18]. He used a combination of the collocation method using quadratic B-splines and the Runge-Kutta method. Dogan applied the Galerkin method to this equation [19]. Also, Zaki used the least square finite element scheme to the EW equation [20]. An spectral method, finite difference method, and cubic B-spline collocation method also has been applied to solve the aforementioned equation. Recently, Yusufoglu and Bekir solved this equation using VIM and ADM [21].

4. Modified Equal-Width Wave Equation

The modified equal-width wave (MEW) equation is formulated as follows:

$$u_t + \varepsilon u^2 u_x - \mu u_{xxt} = 0. \tag{9}$$

This equation has a solitary wave solution of the form

$$u(x,t) = A\ \mathrm{sech}(\frac{1}{\sqrt{\mu}}(x - ct - x_0)), \tag{10}$$

where $A = \sqrt{\frac{6c}{\varepsilon}}$. Authors have used various kinds of numerical methods to solve (9). Zaki [22] used a quintic B-spline collocation method to investigate the motion of a single solitary wave, interaction of two solitary waves, and birth of solitons for the MEW equation. Hamdi et al. [23] derived exact solitary wave solutions of the MEW. Evans and Raslan [24] solved the MEW equation by a collocation finite element method using quadratic B-splines to obtain the numerical solutions of the single solitary wave, solitary waves interaction, and birth of solitons. Also, a linearized numerical scheme based on finite difference method has been used by Esen and Kutluay [25]. Wazwaz [26] investigated the MEW equation and two of its variants by the tanh and the sine-cosine methods.

5. HAM Solutions

In this section we apply HAM to EW and MEW equations. In both problems, we set the initial guess to be $v_0(x,t) = u(x,0)$, i. e. the initial condition, use the auxiliary linear operator $L = \frac{\partial}{\partial t}$ and put $H = 1$ to be the auxiliary function. These simple choices are very effective. To show this fact we will give approximations and compute error terms. All calculations are done with Maple 11, a simple procedure have been written to compute the series terms. Our choices for v_0, L, and H seems valuable, because we don't get use of any rule of solution expression (see [2]) for representing the solution. So our choices are independent of physical backgrounds.

5.1. EW Equation

Considering (7) and (8) we have the zeroth-order deformation equation as follows:

$$(1-q)(\phi_t - v_{0t}) = q\hbar(\phi_t + \phi\phi_x - \phi_{xxt}). \quad (11)$$

Solving the corresponding mth-order deformation equations we have

$$u_0(x,t) = 3\mathrm{sech}^2\left(\frac{x-15}{2}\right),$$

$$u_1(x,t) = -9\mathrm{sech}^4\left(\frac{x-15}{2}\right)\tanh\left(\frac{x-15}{2}\right)\hbar t,$$

$$\begin{aligned}
u_2(x,t) = &\left\{\frac{-189}{4}\mathrm{sech}^8\left(\frac{x-15}{2}\right) + \frac{81}{2}\mathrm{sech}^6\left(\frac{x-15}{2}\right)\right\}\hbar^2 t^2 \\
&+ \left\{9(3\hbar-1)\mathrm{sech}^4\left(\frac{x-15}{2}\right)\tanh\left(\frac{x-15}{2}\right) - \frac{135}{2}\mathrm{sech}^6\left(\frac{x-15}{2}\right)\tanh\left(\frac{x-15}{2}\right)\right\}\hbar t,
\end{aligned}$$

$$\begin{aligned}
u_3(x,t) = &\left\{81\mathrm{sech}^6\left(\frac{x-15}{2}\right) - \frac{189}{2}\mathrm{sech}^8\left(\frac{x-15}{2}\right) + \frac{135}{4}\mathrm{sech}^{10}\left(\frac{x-15}{2}\right)\tanh\left(\frac{x-15}{2}\right)\right. \\
&\left. -24\mathrm{sech}^8\left(\frac{x-15}{2}\right)\tanh\left(\frac{x-15}{2}\right)\right\}\hbar^2 t^2 + \left\{-\frac{99}{2}\hbar\mathrm{sech}^6\left(\frac{x-15}{2}\right) + \frac{747}{4}\hbar\mathrm{sech}^8\left(\frac{x-15}{2}\right)\right. \\
&\left. -\frac{1161}{4}\hbar\mathrm{sech}^{10}\left(\frac{x-15}{2}\right) - 9\mathrm{sech}^4\left(\frac{x-15}{2}\right)\tanh\left(\frac{x-15}{2}\right)\right\}\hbar t \\
&+ \left\{54\mathrm{sech}^4\left(\frac{x-15}{2}\right)\tanh\left(\frac{x-15}{2}\right) - 9\hbar\mathrm{sech}^4\left(\frac{x-15}{2}\right)\tanh\left(\frac{x-15}{2}\right)\right. \\
&-15\mathrm{sech}^6\left(\frac{x-15}{2}\right)\tanh\left(\frac{x-15}{2}\right) - \frac{165}{2}\hbar\mathrm{sech}^6\left(\frac{x-15}{2}\right)\tanh\left(\frac{x-15}{2}\right) \\
&\left. -105\hbar\mathrm{sech}^8\left(\frac{x-15}{2}\right)\tanh\left(\frac{x-15}{2}\right)\right\}\hbar \ldots,
\end{aligned}$$

If we set $\hbar = -1$ in these terms we have exactly the terms obtained by the ADM [20], so we see that the ADM is only a special case of HAM. We use these four terms to construct the approximate solution

$$\mathrm{app}_3(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t). \quad (12)$$

According to the theory of $\hbar$-curves, which is discussed in [2], we plot the $\hbar$-curves corresponding to $u_t(0,0)$, $u_{tt}(0,0)$, and $u_{ttt}(0,0)$, and we have the valid region for $\hbar$ as $R_\hbar = [0.1, 0.5]$. Testing different values of $\hbar$ in this valid region $R_\hbar$, we conclude that the value $\hbar = 0.15$ has the minimum error. We have tabulated the absolute errors of HAM approximation (12) for $\hbar = 0.15$ in Table 1. It is seen that, although we have used only four terms in constructing the approximate solution, it is very close to the exact solution in the time interval discussed in references, see [21].

Table 1. Absolute errors of approximation for EW equation using HAM by $\hbar = 0.15$.

t	$x=0$	$x=5$	$x=10$	$x=15$	$x=20$	$x=25$
0.001	3.66E-9	5.44E-7	8.04E-5	5.52E-8	8.05E-5	5.45E-7
0.002	7.33E-9	1.08E-6	1.60E-4	2.20E-7	1.61E-4	1.09E-6
0.003	1.09E-8	1.63E-6	2.41E-4	4.96E-7	2.41E-4	1.63E-6
0.004	1.46E-8	2.17E-6	3.21E-4	8.83E-7	3.22E-4	2.18E-6
0.01	3.65E-8	5.42E-6	8.01E-4	5.51E-6	8.08E-4	5.47E-6

For EW equation, there are three conservation laws, corresponding to conservation of mass, momentum, and energy; they are [20]:

$$I_1 = \int_a^b u\,\mathrm{d}x, \quad I_2 = \int_a^b (u^2 + u_x^2)\,\mathrm{d}x, \quad I_3 = \int_a^b u^3\,\mathrm{d}x.$$

For computational reasons we chose the interval to be $[0, 80]$, as was chosen in [20]. The efficiency of HAM and its superiority in comparison with VIM and HPM can easily be checked in Table 2, where we have

t	I_1 exact	I_1 HAM	I_1 HPM	I_1 ADM	I_1 VIM
0.001	11.99999633	11.99999633	11.99999633	11.99999637	11.99999636
0.002	11.99999633	11.99999633	11.99999633	11.99999637	11.99999636
0.003	11.99999634	11.99999633	11.99999633	11.99999636	11.99999636
0.004	11.99999634	11.99999633	11.99999633	11.99999636	11.99999636
0.005	11.99999634	11.99999633	11.99999633	11.99999636	11.99999636
0.01	11.99999637	11.99999633	11.99999633	11.99999636	11.99999636
t	I_2 exact	I_2 HAM	I_2 HPM	I_2 ADM	I_2 VIM
0.001	28.80000000	28.80001839	28.80913120	28.80913169	28.80014584
0.002	28.80000000	28.80007357	28.83652503	28.83652549	28.80058191
0.003	28.79999998	28.80016554	28.88218190	28.88218237	28.80130870
0.004	28.80000001	28.80029429	28.94610254	28.94610301	28.80232622
0.005	28.80000000	28.80045984	29.02828799	29.02828846	28.80363453
0.01	28.79999999	28.80183935	29.71324466	29.71324514	28.81453942
t	I_3 exact	I_3 HAM	I_3 HPM	I_3 ADM	I_3 VIM
0.001	57.60000001	57.60004061	57.60863394	57.60863445	57.60009994
0.002	57.60000001	57.60016243	57.63453609	57.63453659	57.60039822
0.003	57.60000000	57.60036546	57.67770728	57.67770778	57.60089544
0.004	57.60000004	57.60064971	57.73814894	57.73814947	57.60159154
0.005	57.60000004	57.60101517	57.81586322	57.81586374	57.60248656
0.01	57.60000001	57.60406070	58.46363371	58.46363419	57.60994790

Table 2. Computed quantities I_1, I_2, and I_3 for EW by HAM, HPM, ADM, and VIM.

Table 3. Absolute errors of approximation for MEW equation using HAM by $\hbar = -2/3$.

t	$x = 20$	$x = 25$	$x = 30$	$x = 35$	$x = 40$	$x = 45$
0.01	2.83E-8	4.202E-6	1.27E-13	4.20E-6	2.83E-8	1.91E-10
0.05	1.41E-7	2.095E-5	1.00E-10	2.10E-5	1.42E-7	9.58E-10
0.1	2.81E-7	4.170E-5	1.20E-9	4.23E-5	2.85E-7	1.92E-9
0.5	1.37E-6	2.038E-4	7.93E-7	2.16E-4	1.46E-6	9.86E-9
1	2.66E-6	3.950E-4	1.26E-5	4.47E-4	3.02E-6	2.03E-8

computed the values of I_1, I_2, and I_3 for different values of t, see [20].

5.2. MEW Equation

Considering (9), we study the case where $\mu = 1$, $\varepsilon = 3$, and $A = 0.25$. Other cases can be treated in a similar way. The initial condition is

$$u(x,0) = \frac{1}{4}\operatorname{sech}(x-30). \tag{13}$$

The zeroth-order deformation equation is constructed as follows:

$$(1-q)(\phi_t - v_{0t}) = q\hbar(\phi_t + 3\phi^2\phi_x - \phi_{xxt}). \tag{14}$$

Solving the corresponding mth-order deformation equation we have

$$u_0(x,t) = \frac{1}{4}\operatorname{sech}(x-30),$$

$$u_1(x,t) = -\frac{3}{64}\operatorname{sech}^3(x-30)\tanh(x-30)\hbar t,$$

$$\begin{aligned} u_2(x,t) = &\left\{\frac{45}{2048}\operatorname{sech}^5(x-30) - \frac{27}{1024}\operatorname{sech}^7(x-30)\right\}\hbar^2 t^2 \\ &+ \left\{-\frac{3}{64}\operatorname{sech}^3(x-30)\tanh(x-30)\right. \\ &+ \frac{3}{8}\hbar\operatorname{sech}^3(x-30)\tanh(x-30) \\ &\left.- \frac{15}{16}\hbar\operatorname{sech}^5(x-30)\tanh(x-30)\right\}\hbar t, \\ &\vdots \end{aligned}$$

We use these three terms to construct the approximate solution

$$\operatorname{app}_3(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t). \tag{15}$$

The maximum error occurs on point $x = 30$, so we search for a value of $\hbar$ which reduces the error at this point (which will consequently reduce the error in other points). From $\hbar$-curves it is seen that $\hbar = -2/3$ is the most suitable value. If one chooses $\hbar = -1$ one will have the HPM results, which is not as good as our choice at the point $x = 30$. Choosing $\hbar = -2/3$ we have tabulated the absolute errors of HAM approximation (15) in Table 3. The points in table are chosen according to [25].

6. Conclusions

In this paper, we have solved the equal-width and modified equal-width wave equations using HAM. We used only four and three terms, respectively, to construct the approximations. These approximations are close enough to the exact solutions as can be easily checked in Tables 1, 2, and 3. The results are valuable because we have a continuous approximation, which is useful for computational purposes.

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